ANALYTICAL GEOMETRY (3D) AND INTEGRAL CALCULUS

UNIT I

Standard equation of a plane – intercept form-normal form-plane passing through given points – angle between planes –plane through the line of intersection of two planes- Equation of the straight line – Shortest distance between two skew lines- Equation of the line of shortest distance

UNIT II

Sphere – Standard equation –Length of a tangent from any point-Sphere passing through a given circle – intersection of two spheres – Tangent plane.

UNIT III

Integration by parts – definite integrals & reduction formula

UNIT IV

Double integrals – changing the order of Integration – Triple Integrals.

UNIT V

Beta & Gamma functions and the relation between them –Integration using Beta & Gamma functions

TEXT BOOK(S)

[1] T.K.Manickavasagam Pillai & others, Analytical Geometry, S.V Publications -1985 Revised Edition.
[2] T.K.Manickavasagam Pillai & others, Integral Calculus, SV Publications.

UNIT – I - Chapter 2 Sections 13 to 21 & Chapter 3 Sections 24 to 31 of [1]
UNIT – II - Chapter 4 Sections 35 to 42 of [1]
UNIT – III - Chapter 1 Sections 11 , 12 & 13 of [2]
UNIT – IV - Chapter 5 Sections 2.1 , 2.2 & Section 4 of [2]
UNIT – V - Chapter 7 Sections 2.1 to 5 of [2]

REFERNECE(S)

[1] Duraipandian and Chatterjee, Analytical Geometry
[2] Shanti Narayan, Differential & Integral Calculus.

PART – A

1. Write down the formula to find the angle between Planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

The angle between two planes is

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

2. Write down the Condition for the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

to be Parallel to the Plane $ax + by + cz + d = 0$

Condition for the line to be parallel to the plane is

$$al + bm + cn = 0 \;\; \textbf{and} \;\; ax_1 + by_1 + cz_1d \neq 0$$

3. Formula for the Length of the tangent from the Point (X, Y, Z)tothe Sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is ?

The length of the tangent from a point to the sphere is

$$PT = \sqrt{x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d}$$

4. Writethe formula for equation of Tangent plane to Sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is ? at (x, y, z)

The equation of tangent plane to sphere is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

5. Write down the intercept form of the equation of a plane

The intercept form of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

6. Write down the Condition for two given straight lines to be coplanar.

The condition for two lines

$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ *and* $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ to be coplanar is

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 + z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

7. Find the centre and radius of the Sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

The centre is $c(-u,-v,-w)$

The radius $r = \sqrt{u^2 + v^2 + w^2 - d}$

8. Write down the formula for intersection of two spheres:

The intersection of two sphere is

$$s_1 - s_2 = 2x(u - u_1) + 2y(v - v_1) + 2z(w - w_1) + d - d_1 = 0$$

9. Find the equation of the plane through Pt(2, -4, 5) and parallel to the plane

$4x - 2y - 7z + 6 = 0$

Equation of a plane parallel to $4x - 2y - 7z + 6 = 0$ *is* $4x - 2y - 7z + K = 0$

Given 1 passes thro $(2, -4, 5)$

$$=> 4(2) - 2(-4) - 7(5) + K = 0$$

$$8 + 8 - 35 + K = 0$$

$$\boldsymbol{K = 19}$$

$$Sub\ K = 19\ in\ 1$$

$$\boldsymbol{The\ equation\ of\ the\ plane\ is\ 4x - 2y - 7z + 19 = 0}$$

10. Find the point where the line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z+2}{5}$

meets the plane $x - y + z = 5$

Let the given line be

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z+2}{5} = r$$

$$\therefore \text{any point on the line is } (3r+2, 4r+3, 5r-2)$$

$$\text{it meets the plane } x - y - z = 5$$

$$=> (3r+2) - (4r+3) + (5r-2) - 5 = 0$$

$$3r - 4r + 5r + 2 - 3 - 2 - 5 = 0$$

$$4r = 8$$

$$r = {}^{8}/_{4} => \; r = 2$$

$$\therefore \boldsymbol{the\ point\ of\ contact\ is (8, 11, 8)}$$

11. Find the Centre and Radius of the Sphere $\boldsymbol{4(x^2+y^2+z^2)+10x-25y-2z=0}$

Given

$$S: 4(x^2+y^2+z^2) + 10x - 25y - 2z = 0$$

$$=> x^2 + y^2 + z^2 + \frac{10}{4}x - \frac{25}{4}y - \frac{2}{4}z = 0$$

$\therefore \boldsymbol{Centre, C\left(\frac{-5}{4}, \frac{25}{8}, \frac{1}{4}\right)}$

Radius $r = \sqrt{\left({}^{-5}/_{4}\right)^2 + \left({}^{25}/_{8}\right)^2 + \left({}^{1}/_{4}\right)^2 - 0}$

$$= \sqrt{\frac{25}{16} + \frac{625}{64} + \frac{1}{16}} = \sqrt{\frac{\cancel{26}\,13}{\cancel{16}\,8} + \frac{625}{64}}$$

$$= \sqrt{\frac{104+625}{64}} = \sqrt{\frac{729}{64}} = \frac{27}{8} \quad \boldsymbol{Radius\ \ r = \frac{27}{8}}$$

12. Find the equation of the Sphere thro the circle $\boldsymbol{x^2+y^2+z^2=9}$,

$2x + 3y + 4z = 5$ **and point (1,2,3)**

Given : $s = x^2 + y \;\; + z^2 - 9 = 0$ **;** $2x + 3y + 4z - 5 = 0$

The equation of sphere thro circle is $S + KL = 0$

$$=> (x^2 + y^2 + z^2 - 9) + K\,(2x + 3y + 4z - 5) = 0$$

1 passes thro (1,2,3)

$$(1 + 4 + 9 - 9) + K(\,2(1) + 3(2) + 4(3) - 5) = 0$$

$\boldsymbol{K = {}^{-1}\!/_{3}}$ **;** *Subs K in* 1

$$\boldsymbol{The\ equation\ of\ sphere\ is\ \ 3x^2 + 3y^2 + 3z^2 - \ 2x - 3y - 4z - 22 = 0}$$

13. Write the equation of the plane making intercepts $\boldsymbol{2, 3, 4}$ *on the axes*

ox, oy, oz respectively

Given a=2,b=3,c=4

Equation of plane in intercept form is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$=> \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$$

$$6x + 4y + 3z = 12$$

$$\boldsymbol{The\ equation\ of\ the\ plane\ is\ \ 6x + 4y + 3z - 12 = 0}$$

14. Find the length of the Tangent from the point $\boldsymbol{(1, 0\,, 2)}$to the Sphere

$$\boldsymbol{x^2 + y^2 + z^2 + 4x + 2y + 6z + 3 = 0}$$

GIVEN : S: $\quad x^2 + y^2 + z^2 - 4x + 2y + 6z + 3 = 0 \;\; at \;\; (1, 0, 2)$

The Length of The Tangent is

$$PT^2 \coloneqq \; x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

$$=> \; PT^2 = (1)^2 + (0)^2 + (2)^2 + (-4)1 + 2(0) + 6(2) + d$$

$$= 5 - 4 + 15 = 16$$

$$=> \; PT^2 = 16$$

$$=> PT = 4$$

$$\boldsymbol{LENGTH\ OF\ THE\ TANGENT}, \quad \boldsymbol{PT = 4 units}$$

15. Write down the Equation of the Plane $\boldsymbol{\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}}$

and Parallel to the line $\boldsymbol{\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}}$

The Equation of the Plane Containing the Line

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \&$$

$$\|^{ly}\ TO \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}\ is\,,$$

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$\boldsymbol{together\ \ with \quad al + bm + cn = 0 \ \ \&\ \ al_1 + bm_1 + cn_1 = 0}$$

16. What are the Intercepts of $\boldsymbol{ax + by + cz + d = 0}$?

$$ax + by + cz + d = 0$$

$$ax + by + cz = -d$$

$$\div\ by\ (-d) => \frac{a}{-d}x + \frac{b}{-d}y + \frac{c}{-d}z = 1$$

$$=> \frac{x}{-d/a} + \frac{y}{-d/b} + \frac{z}{-d/c} = 1$$

$$\boldsymbol{\therefore x\ intercepts\ is\ \ {-d}/{a}}$$

$$\boldsymbol{\therefore y\ intercepts\ is\ \ {-d}/{b}}$$

$$\boldsymbol{\therefore z\ intercepts\ is\ \ {-d}/{c}}$$

17. Find the Length of Perpendicular drawn from (1, 2, 3) on $\boldsymbol{2x + 3y + 4z + 8 = 0}$

Given:

$$2x + 3y + 4z + 8 = 0 \quad => \quad (x_1, y_1, z_1) = (1, 2, 3)$$

$$length\ of\ \perp r\ drawn\ from (1, 2, 3) on\ plane$$

$$= \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$= \left| \frac{2(1) + 3(2) + 4(3) + 8}{\sqrt{2^2 + 3^2 + 4^2}} \right| = \left| \frac{2 + 6 + 12 + 8}{\sqrt{4 + 9 + 16}} \right|$$

$$= \left| \frac{2 + 6 + 12 + 8}{\sqrt{4 + 9 + 16}} \right| = \left| \frac{28}{\sqrt{29}} \right|$$

$$\boldsymbol{length\ of\ \perp r\ drawn = \frac{28}{\sqrt{29}} units}$$

18. Find the Length of the Tangent $(\mathbf{1}, -\mathbf{1}, \mathbf{1})$ to the Sphere $\boldsymbol{x^2 + y^2 + z^2 + k = 0}$

Given :

$$S: \ x^2 + y^2 + z^2 + k = 0 \qquad => \quad (x_1 + y_1 + z_1) = (1, -1, 1)$$

$$length\ of\ the\ tangent\ is \quad PT^2 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

$$PT^2 = 1^2 + (-1)^2 + (1)^2 + K$$

$$PT^2 = 3 + K$$

$$PT = \sqrt{3 + K}$$

$$\boldsymbol{length\ of\ the\ tangent\ is\ Pt = \sqrt{K + 3} units}$$

19. State the Condition for the Straight Line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

to lie on the plane $ax + by + cz + d = 0$

Condition for line to lie on the plane:

The line should be parallel to the plane and

any point on the line should pass through the plane.

20. Find the tangent plane at (-1, 4, -2) on $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$

The equation of the tangent plane is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

The equation of tangent plane at (-1, 4,-2) on $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ is

$$x(-1) + y(4) + z(-2) + (-2)\left(\frac{x+(-1)}{2}\right) - 4\left(\frac{y+4}{2}\right) + 2\left(\frac{z-2}{2}\right) - 3 = 0$$

$$-x + 4y - 2z - x + 1 - 2y - 8 + z - 2 - 3 = 0$$

$$-2x + 2y - z - 12 = 0$$

$$2x - 2y + z + 12 = 0$$

The equation of the tangent plane is $2x - 2y + z + 12 = 0$

21. Find the normal form of $2x - 6y + 3z = 2$.

Given :

$$2x - 6y + 3z - 2 = 0$$

$$\Rightarrow a = 2; b = -6; c = 3; d = 2$$

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{2}{\sqrt{2^2+(-6)^2+3^2}} = \frac{2}{\sqrt{4+36+9}} = \frac{2}{\sqrt{49}}$$

$$\boldsymbol{l = \frac{2}{7}}$$

$$similarly\ \ m = \frac{-6}{7}\ \ and\ \ \ n = \frac{3}{7}$$

$$normal\ form\ is\ lx + my + nz = p$$

$$p = \frac{d}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore\ \ \boldsymbol{p = \frac{2}{7}}$$

$$The\ normal\ form\ is\ \frac{2}{7}x - \frac{6}{7}y + \frac{3}{7}z = \frac{2}{7}$$

$$=>\ 2x - 6y + 3z = 2.$$

$$\boldsymbol{The\ normal\ form\ is\ \ 2x - 6y + 3z - 2 = 0}$$

22. Evaluate:

$$\int_{-\pi/4}^{\pi/4} \int_{0}^{a\sqrt{\cos 2\theta}} \boldsymbol{r dr\, d\theta}$$

Solution:

$$\int_{-\pi/4}^{\pi/4} \int_{0}^{a\sqrt{\cos 2\theta}} r dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left[\int_{0}^{a\sqrt{\cos 2\theta}} r dr \right] d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left(r^2/2 \right)_{0}^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4} \frac{a^2\ cos2\theta}{2}\ d\theta$$

$$= {a^2}/{2} \int_{-\pi/4}^{\pi/4} a^2 \, cos2\theta \quad d\theta$$

$$= a^2 \left[\frac{sin2\theta}{2}\right]_0^{\pi/4}$$

$$\int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \boldsymbol{rdr \, d\theta = {a^2}/{2}}$$

23. Evaluate:

$$\int_0^{\pi/2} \boldsymbol{cos^7\theta \;\; cos^5\theta \;\; d\theta}$$

Solution

$$\boldsymbol{W.K.T} \int_0^{\pi/2} cos^u\theta \;\; cos^q\theta \;\; d\theta = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+1}{2}}$$

$$\int_0^{\pi/2} cos^7\theta \;\; cos^5\theta \;\; d\theta = \frac{\Gamma 4 \Gamma 3}{2\Gamma 7}$$

$$= \frac{3! \times 2!}{2 \times 6!}$$

$$= \boldsymbol{\frac{1}{120}}$$

24. Write down the recurrence formula for Γn

Solution:

Recurrence formula for Γn

$$\Gamma n + 1 = n\Gamma n \;\; , \qquad n > 0; which\ is\ true\ for\ only\ n > 0$$

25. . Evaluate:

$$\int_0^\infty \frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{x})\boldsymbol{x}^{4/3}}\,\boldsymbol{dx}$$

Solution:

$$\int_0^\infty \frac{1}{(1+x)x^{4/3}}\,dx = \int_0^\infty \frac{x^{-3/4}}{(1+x)}\,dx = \int_0^\infty \frac{x^{1/4-1}}{1+x}\,dx$$

$$\therefore \int_0^\infty \frac{1}{(1+x)x^{3/4}}\,dx = \frac{\pi}{sin(1/4)\pi}$$

$$= \frac{\pi}{1/\sqrt{2}}$$

$$\therefore \int_0^\infty \frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{x})\boldsymbol{x}^{3/4}}\,\boldsymbol{dx} = \sqrt{\mathbf{2}}\,\boldsymbol{\pi}.$$

26. Evaluate

$$\int_0^1 \boldsymbol{x}^{\mathbf{4}}(\mathbf{1}-\boldsymbol{x})^{\mathbf{3}}\ \boldsymbol{dx}$$

Solution:

$$\boldsymbol{W.K.T.}\int_0^1 x^{m-1}(1-x)^{n-1}\ dx = \beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

$$\int_0^1 \boldsymbol{x^4(1-x)^3\ dx} = \frac{\boldsymbol{\Gamma 5 \Gamma 4}}{\boldsymbol{\Gamma 9}} = \frac{\boldsymbol{4!\ \ 3!}}{\boldsymbol{8!}} = \boldsymbol{{}^{1}/_{280}}$$

27. Find the value of Γn

Solution:

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\ dx$$

$$\Gamma 1 = \int_0^\infty e^{-x} x^{0}\ dx$$

$$= [-e^{-x}]_0^\infty\ =\ -[0-1]$$

$$\boldsymbol{\Gamma 1 = 1}$$

28. Find the value of Γ $\frac{-1}{2}$

Solution: $\Gamma n = \frac{\Gamma n+1}{n}$

$$\Gamma \frac{-1}{2} = \frac{\Gamma \frac{-1}{2}+1}{{}^{-1}/_{2}} = \ -2\Gamma \frac{-1}{2}$$

$$\Gamma \frac{-1}{2} = \boldsymbol{-2\sqrt{\pi}.}$$

29. Evaluate:

$$\int_0^\infty \boldsymbol{\sqrt{x}\, e^{-x^3}\ dx}$$

$$put\ \ t = x^3 \quad => x = t^{1/3}$$

$$dt = 3x^2 dx$$

$$dx = \frac{dt}{3x^2}$$

$$when\ x = 0,\ \ t = 0\ ;\ \ \ x = \infty, \qquad t = \infty$$

$$\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx \ = \int_0^\infty \sqrt{t^{1/3}}\ e^{-t}\ \frac{dt}{3t^{2/3}}$$

$$= {}^1/_3 \int_0^\infty t^{1/6}\ e^{-t}\ \ t^{-2/3}\, dt$$

$$= {}^1/_3 \int_0^\infty t^{-3/6}\ e^{-t}\ \ dt$$

$$= {}^1/_3 \int_0^\infty t^{-1/2}\ e^{-t}\ \ dt$$

$$= {}^1/_3 \sqrt{-{}^1/_2 + 1}$$

$$= {}^1/_3 \sqrt{{}^1/_2}$$

$$= \frac{\sqrt{\boldsymbol{\pi}}}{\mathbf{3}}$$

30. $\boldsymbol{Define\ (\beta)\ beta\ function}$

$Beta\ function\ can\ be\ defined\ as\ \ \beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1} dx$

31. Write down the formula for relation between beta and gamma function

$The\ formula\ for\ relation\ between\ (\beta) beta\ and\ (\gamma) gamma\ function\ is$

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m + n} \quad for\ m > 0, \qquad n > 0$$

11. Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$\boldsymbol{W.K.T} \quad \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n} \quad Here \quad m = n = {}^{1}/_{2}$$

$$\beta({}^{1}/_{2}, {}^{1}/_{2}) = \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}+\frac{1}{2}}$$

$$= \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1} \quad Since \;\; \Gamma 1 = 1! = 1$$

$$\beta({}^{1}/_{2}, {}^{1}/_{2}) = \left(\Gamma\frac{1}{2}\right)^2$$

$$W.K.T \quad \beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \;\; cos^{2n-1}\theta \, d\theta$$

$$Here \quad m = n = {}^{1}/_{2}$$

$$\beta({}^{1}/_{2}, {}^{1}/_{2}) = 2\int_0^{\pi/2} sin^{2({}^{1}/_{2})-1}\theta \;\; cos^{2({}^{1}/_{2})-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} sin^{\circ}\theta \;\; cos^{\circ}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2 \cdot {}^{\pi}/_{2}$$

$$\beta({}^{1}/_{2}, {}^{1}/_{2}) = \pi$$

From 1 and 2

$$\left(\Gamma\frac{1}{2}\right)^2 = \pi$$

$$\Gamma\frac{1}{2} = \sqrt{\pi}.$$

12. $\boldsymbol{\beta(m,n) = \beta(n,m)}$

Solution:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$$

By the property of definite integrals

$$\int_0^a f(x)\ dx = \int_0^a f(a-x)dx$$

$$\beta(m,n) = \int_0^1 (1-x)^{m-1}x^{n-1}\ dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\ dx$$

$$\boldsymbol{\beta(m,n) = \beta(n,m)}$$

13. Find the value of $\Gamma 0$ *and* $\Gamma 1 - n$ *where* 'n' is a positive integer.

(i) $\Gamma 0$

$\boldsymbol{W.K.T}$ $\Gamma n + 1 = n\Gamma n$

$$\Gamma n = \frac{\Gamma n + 1}{n}$$

$$Here\ \ n = 0, \qquad \Gamma 0 = \frac{\Gamma 0 + 1}{0}$$

$$= \frac{\Gamma 1}{0}$$

$$\Gamma 0 = \frac{1}{0}$$

$$\Gamma 0 = \infty.$$

$$\therefore\ \Gamma \mathbf{0}\ \boldsymbol{is\ undefined}$$

$$(ii)\ \ \Gamma 1 - n$$

$$\Gamma 1 - n = (1 - n - 1)\Gamma 1 - n - 1$$

$$= -n\Gamma - n\ = undefined$$

$$Since\ \ \Gamma - n\ \ is\ \ undefined$$

$$\Gamma n \Gamma 1 - n = \frac{\pi}{sin\, n\pi}$$

$$\Gamma 1 - n = \frac{1}{\Gamma n}\ \cdot \frac{\pi}{sin\, n\pi}$$

14. State the recurrence formula for gamma function

$The\ recurrence\ formula\ for\ gamma\ function\ is$ $\Gamma n + 1 =\ n\Gamma n\ \ \ (or)\ \ \Gamma n = \frac{\Gamma n+1}{n}$.

PART – B

1. Find the shortest distance

$$\frac{x-3}{3} = 8 - y = z - 3;\ \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Solution:

Let $\frac{x-3}{3} = \frac{8-y}{1} = \frac{z-3}{1} = r_1$ ⟶ ①

$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = r_1$ ⟶ ②

Any point on line ① is $G(3r_1+3, -r_1+8, r_1+3)$

Any point on line ② is $H(-3r_2-3, 2r_2-7, 4\,r_2+6)$

The direction of line joining the point G & H

$$(3r_1+3+3r_2+3, -r_1+8-2r_2+7, r_1+3-4r_2-6)$$

$$(ie)[\ (3r_1+3r_2+6), (-r_1-2r_2+15), (\ r_1-4r_2-3)]$$

The direction's of Line ① (3,-1,1)

$$Since\ GH\ \ us\ \perp r\ to\ AB\ by\ perpendicular\ condition\ l_1l_2+m_1m_2+n_1n_2=0$$

We've $9r_1+9r_2+18+r_1+2r_2-15+r_1-4r_2-3=0$

$11r_1+7r_2=0$ ⟶ ③

Also the direction of line ② is (-3,2,4)

$$GH\ is\ perpendicular\ to\ ②$$

We've $-9r_1-9r_2-18-2r_1-4r_2-30+4r_1-16r_2-12=0$

$$-7r_1 - 29r_2 = 0 \longrightarrow ④$$

Solving ③ & ④

③ X 7 => $77r_1 + 49r_2 = 0$

④ X 11=> $-77r_1 + 319r_2 = 0$

$-270r_2 = 0$

$$\boldsymbol{r_2 = 0}$$

Sub $r_2 = 0$ in ③

$$11r_1 + 0 = 0$$

$$11r_1 = 0$$

$$r_1 = 0$$

$sub\ \ r_1 = 0\ \ \& \, r_2 = 0\ in\ pt\ G\ \&\ H$

$$\therefore G\ \ Co - ordinates\ are\ (3,8,3)$$

$$\therefore H\ Co - ordinates\ are(-3,-7,6)$$

Shortest distance between G & H is

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2}$

$= \sqrt{(-6)^2 + (-15)^2 + (3)^2}$

$= \sqrt{36 + 225 + 9} = \sqrt{270}$

$\therefore$ ***shortest distance between given the skew line is***

$$= \sqrt{270}$$

2. Find the centre and radius of the circle $x^2 + y^2 + z^2 - 2y - 4z = 11$ *and*

$x + 2y + 2z = 15$

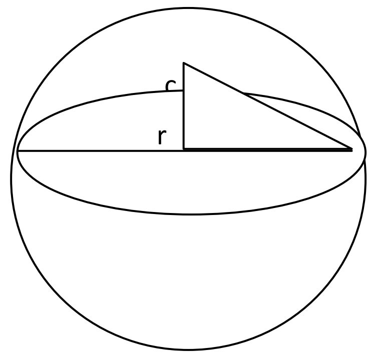
Solution:

Given the equation of the sphere is $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$

$centre = (-u, -v, -w)$

$centre = (0,1,2)$

$radius = \sqrt{0 + 1 + 4 + 11} = \sqrt{16} = 4$



$Let\ the\ foot\ of\ the\ \perp r\ Q from\ the\ centre\ (0,1,2) to\ the\ plane\ x + 2y + 2z - 15 = 0\ by\ (x, y, z)$

The direction of CQ is (x-0,y-1,z-2)

The direction of $x + 2y + 2z - 15 = 0$ is $(1,2,2)$

The line CQ is $\perp r$ to the given plane and hence parallel to the normal to the plane

The direction of proportional

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{2} = K$$

The point Q is (K, 2K+1,2K+2)

The point is lies on the plane $x + 2y + 2z - 15 = 0$

$$K + 2(2K + 1) + 2(2K + 2) - 15 = 0$$

$$K + 4K + 2 + 4K + 4 - 15 = 0$$

$$9K - 9 = 0$$

$$9K = 9$$

$$K = \frac{9}{9} = 1$$

$$\boldsymbol{K = 1}$$

The point Q is (1,3,4)

Distance between two points $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$CQ = \sqrt{(1-0)^2 + (3-1)^2 + (4-2)^2}$$

$$= \sqrt{1+4+4} = \sqrt{9} = 3$$

$$CQ = 3\, units$$

Radius of the circle $QP = \sqrt{r^2 - CQ^2}$

$$QP = \sqrt{4^2 - 3^2} = \sqrt{16-9} = \sqrt{7}$$

$$radius\ of\ the\ circle\ QP = \sqrt{7}\, units$$

$$\boldsymbol{Centre\ of\ the\ circle\ is\ (1,3,4)\ and\ radius\ is\ \sqrt{7}\ units}$$

3. Derive the intercept form of a plane

Statement :

find the equation of a plane making intercepts a,b.c on the axis ox, oy, oz respectively

Proof:

let the given plane meet the co-ordinate axis ox, oy, oz, at A, B, C respectively

$$\therefore OA = a$$

$$\therefore OB = b$$

$$\therefore OC = c$$

Hence the co-ordinates of the points A,B,C are respectively (a,0,0) , (0,b,0), (0,0,c)

$$Let\ the\ equation\ of\ the\ plane\ be, Px + Qy + Rz + S = 0$$

$Since\ it\ passes\ through\ the\ point\ A, B, C\ We've$

Subs the values in

$\frac{-S}{a}x + \left({}^{-s}/_{b}\right)y + \left({}^{-S}/_{c}\right)z + S = 0$

$(ie) - S\left[{}^{x}/_{a} + {}^{y}/_{b} + {}^{z}/_{c} - 1\right] = 0$

$(ie)\, {}^{x}/_{a} + {}^{y}/_{b} + {}^{z}/_{c} - 1 = 0$

${}^{x}/_{a} + {}^{y}/_{b} + {}^{z}/_{c} = 1$

This is known as intercept form of the equation of a plane.

4. Find the equation of the plane through the line
$\boldsymbol{\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+1}{2}}$ ***and parallel to***

$\boldsymbol{\frac{x-2}{2} = \frac{1-y}{3} = \frac{z+4}{3}}$

Solution:

Given that

$$\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+1}{2} \longrightarrow ①$$

Equation of the plane passing through line ① is

$$a(x-1) + b(y+6) + c(z+1) = 0 \longrightarrow ②$$

$$3a + 4b + 2c = 0 \longrightarrow ③$$

Since it is parallel to the line

$$\frac{x-2}{2} = \frac{1-y}{3} = \frac{z+4}{3} \quad we've\ 2a - 3b + 3c = 0 \longrightarrow ④$$

Solving ③ & ④

$$\frac{a}{12+6} = \frac{b}{4-9} = \frac{c}{-9-8}$$

$$\frac{a}{18} = \frac{b}{-5} = \frac{c}{-17}$$

$$a = 18;\ b = -5;\ c = -17$$

$$sub\ a, b, c\ in$$

$$18(x-1) - 5(y+6) - 17(z+1) = 0$$

$$18x - 18 - 5y - 30 - 17z - 17 = 0$$

$$\mathbf{18x - 5y - 17z - 65 = 0}\ \textit{which is a required equation of the plane}.$$

5. Obtain the equation of the sphere through the origin and the circle given by

$$\boldsymbol{x^2 + y^2 + z^2 = 1, x^2 + y^2 + z^2 + x + 2y + 3z - 5 = 0}$$

Solution:

Equation of the sphere passing through the given circle is

$$(x^2 + y^2 + z^2 - 1) + K(x^2 + y^2 + z^2 + x + 2y + 3z - 5) = 0 \longrightarrow ①$$

$$x^2 + y^2 + z^2 - 1 + Kx^2 + Ky^2 + Kz^2 + Kx + K2y + K3z - 5K = 0 \longrightarrow ②$$

Since it passes through the origin (0,0,0)

$-1-5K=0; \quad K=\frac{-1}{5}$

$sub\ \ K=\frac{-1}{5}\ in$ ①

$$x^2+y^2+z^2-1+\left(-1/_5\right)(x^2+y^2+z^2+x+2y+3z-5)=0$$

$$5x^2+5y^2+5z^2-5-x^2-y^2-z^2-x-2y-3z+5=0$$

$$4x^2+4y^2+4z^2-x-2y-3z=0$$

$$4(x^2+y^2+z^2)-x-2y-3z=0$$

Which is a required equation of the sphere

6. ***Show that the plane*** $\boldsymbol{2x-2y+z+12=0}$ ***touches the sphere***

$\boldsymbol{x^2+y^2+z^2-2x-4y+2z-3=0}$ ***And find the point the contact***

Solution:

Given $x^2+y^2+z^2-2x-4y+2z-3=0$

Centre=(-u, -v, -w)

Centre= (1, 2, -1)

$radius=\ \sqrt{u^2+v^2+w^2-d}$

$=\ \sqrt{1+4+1+3}=\sqrt{9}=3\ units$

The perpendicular distance from the centre to the plane $2x-2y+z+12=0\ is$

$$perpendicular\ distance = \frac{2(1) - 2(2) + (-1) + 12}{\sqrt{4+4+1}} = \frac{9}{3}$$

Perpendicular distance =3 units

From ①&②

The $\perp r$ *distance from the centre* $(1,2,-1)$ *of the sphere to the plane*

$$2x - 2y + z + 12 = 0 \text{ is equal to the radius of the sphere.}$$

Thus the given plane touch the sphere let the point of contact be $p(x, y, z)$,

direction ratio's of the line joining the centre $(1, 2, -1)$ *of the sphere and*

$p(x, y, z)$ *are* $(x-1, y-2, z+1)$ *and direction ratio's of the normal to the plane*

$$2x - 2y + z + 12 = 0 \text{ are parallel}$$

We've

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = r$$

Any point on this line is $(2r+1, -2r+2, r-1)$ ⟶ ③

Since this point lies on the plane $2x - 2y + z + 12 = 0$

$$2(2r+1) - 2(-2r+2) + r - 1 + 12 = 0$$

$$4r + 2 + 4r - 4 + r - 1 + 12 = 0$$

$$9r = -9$$

$$\boldsymbol{r = \frac{-9}{9} = -1}$$

Sub $r = -1$ *in* 3 *we've*

$$2(-1) + 1, -2(-1) + 2, -1 - 1$$

$$(\boldsymbol{ie})\ (-\mathbf{1}, \mathbf{4}, -\mathbf{2})$$

$$\therefore\ \boldsymbol{the\ point\ of\ contact\ of\ the\ sphere}\ \&\ plane\ is\ p(-1, 4, -2).$$

7. Prove $\int_0^{\pi/2} \log(\sin x)\, dx = \frac{\pi}{2} \log\left({}^{1}/_{2}\right)$

Solution:

$$I = \int_0^{\pi/2} \log(sin\ x)\, dx$$

$$Put\ x = \frac{\pi}{2} - x$$

$$I = \int_0^{\pi/2} \log\left(sin\ \left({}^{\pi}/_{2} - x\right)\right) dx$$

$$I = \int_0^{\pi/2} \log(cos\ x)\, dx$$

Adding 1 & 2 we get

$$2I = \int_0^{\pi/2} \log(sin\ x)\, dx + \int_0^{\pi/2} \log(cos\ x)\, dx$$

$$2I = \int_0^{\pi/2} \log(sin\ x\ cos\ x)\, dx$$

$$2I = \int_0^{\pi/2} \log\left(\frac{sin\ 2x}{2}\right) dx$$

$$2I = \int_0^{\pi/2} (\log\ \sin 2x - log 2)\, dx$$

$$2I = \int_0^{\pi/2} \log \sin 2\mathrm{x}\, dx - \int_0^{\pi/2} \log 2\, dx$$

$$2I = \int_0^{\pi/2} \log \sin 2\mathrm{x}\, dx - {}^{\pi}/_{2} \log 2$$

To evaluate $\int_0^{\pi/2} \log \sin 2\text{x}\, dx$

$Put\ 2x = y$

$2dx = dy$

$\frac{dy}{2} = dx$

$when\ x = 0, y = 0; x = \frac{\pi}{2}, y = \pi$

$$\int_0^{\pi/2} \log \sin 2x\, dx = \int_0^{\pi} \log \sin \text{y}\ \frac{\text{dy}}{2}$$

$$= {}^1/_2 \int_0^{\pi} \log \sin \text{y}\, dy$$

$$= {}^1/_2 \cdot 2 \int_0^{\pi/2} \log \sin \text{y}\, dy$$

$$= \int_0^{\pi/2} \log \sin \text{y}\, dy$$

$$= \int_0^{\pi/2} \log \sin \text{x}\, dx\ by\ using\ the\ properly$$

Sub 4 in 3 we get

$$2I = \int_0^{\pi/2} \log \sin \text{x}\, dx - {}^{\pi}/_2 \log 2$$

$$2I = I - {}^{\pi}/_2 \log 2$$

$$2I - I = -{}^{\pi}/_2 \log 2$$

$$I = -{}^{\pi}/_2 \log 2$$

$$I = {}^{\pi}/_2 \log\ (2)^{-1}$$

$$I = {}^{\pi}/_2 \log\ ({}^1/_2)$$

$$\therefore \int_0^{\pi/2} \log(\sin x)\, dx = \pi/2 \log(1/2)$$

8. Evaluate

$$\int \boldsymbol{log(a^2 - x^2)\, dx}$$

Solution:

$$u = log(a^2 - x^2) \quad du = \frac{1}{a^2 - x^2}(-2x)dx$$

$$\int dv = \int dx$$

$$v = x$$

$$\int udv = uv - \int vdu$$

$$\int log(a^2 - x^2)\ dx = log(a^2 - x^2)x - \int x\frac{1}{a^2 - x^2}(-2x)dx$$

$$= x\ log(a^2 - x^2) + \int \frac{2x^2}{a^2 - x^2}dx$$

$$= x\ log(a^2 - x^2) + 2\int \frac{a^2(a^2 - x^2)}{a^2 - x^2}dx$$

$$= x\ log(a^2 - x^2) + 2\left[\int \frac{a^2dx}{a^2 - x^2} - \int \frac{a^2 - x^2}{a^2 - x^2}dx\right]$$

$$= x\ log(a^2 - x^2) + 2\ \left(a^2\int \frac{dx}{a^2 - x^2} - \int dx\right)$$

$$= x\ log(a^2 - x^2) + 2a^2\left[\frac{1}{2a}\ log\left(\frac{a + x}{a - x}\right) - x\right]$$

$$\int \boldsymbol{log(a^2 - x^2)\ dx = x\ log(a^2 - x^2) + \ a\ log\left(\frac{a + x}{a - x}\right) - x}$$

9. Evaluate

$$\int_0^1 \boldsymbol{\sin^{-1} x\ dx}$$

$\sin^{-1} x = u$

$$du = \frac{1}{\sqrt{1-x^2}} dx$$

$$\int dv = \int dx$$

$v = x$

$$\int_0^1 \sin^{-1} x \; dx = \sin^{-1} x \cdot x \int_0^1 - \int_0^1 x \cdot \frac{1}{\sqrt{1-x^2}} \; dx$$

$$= \frac{\pi}{2} + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \; dx$$

$$= \frac{\pi}{2} + \frac{1}{2} \left(2\sqrt{1-x^2}\right)_0^1 \quad \left[since \int \frac{f\left(1/x\right)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)}\right]$$

$$= \frac{\pi}{2} + \frac{1}{2}(2)(0-1)$$

$$= \boldsymbol{\frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1}$$

10. Evaluate :

$$\boldsymbol{\int \frac{x + \sin x}{1 + \cos x} dx}$$

Solution:

WKT

$$\sin\theta = \frac{2tan\, \theta/2}{1 + tan^2\, \theta/2}$$

$$\cos\theta = \frac{1 - tan^2 \theta/2}{1 + tan^2 \theta/2}$$

$$\int \frac{x + \sin x}{1 + \cos x} dx = \int \frac{x + \frac{2\tan x/2}{1+tan^2 x/2}}{1 + \frac{1-tan^2 x/2}{1+tan^2 x/2}} \, dx$$

$$= \int \frac{x + xtan^2 x/2 + 2\, tan\, x/2}{1 + tan^2 x/2 + 1 - tan^2 x/2} dx$$

$$= \frac{1}{2}\int \left(x\left(1 + tan^2 x/2\right) + 2\, tan\, x/2\right) dx$$

$$= \frac{1}{2}\left[\int x\, sec^2 x/2\, dx \; + 2\int \; tan\, x/2\, dx\right]$$

Put $t = tan\, x/2 => 2tan^{-1}(t) = x$

$dt = sec^2 x/2 \cdot dx/2 => 2dt = sec^2 x/2\, dx$

$$dx = \frac{2dt}{1 + t^2}$$

$$= \frac{1}{2}\left[\int 2tan^{-1}(t)2dt \; + 2\int \frac{t2dt}{1 + t^2}\right]$$

$$= \frac{1}{2}\left[4\int tan^{-1}(t)dt \; + 4\int \frac{t}{1 + t^2} dt\right]$$

$$= \frac{4}{2}\left[\int tan^{-1}(t)dt \; + \int \frac{tdt}{1 + t^2}\right]$$

$$= 2\left[\int tan^{-1}(t)dt \; + \int \frac{tdt}{1 + t^2}\right]$$

$I = 2[I_1 + I_2]$

$$I_1 = \int tan^{-1}(t)dt$$

$$u = tan^{-1}(t)\, ; dv = dt$$

$$du = \frac{dt}{1+t^2} \;\; ; \;\; v = t$$

$$I_1 = tan^{-1}(t) \cdot t - \int t \cdot \frac{dt}{1+t^2}$$

$$since\ I_2 = \int \frac{dt}{1+t^2}$$

$$\therefore I = 2\left[t\, tan^{-1}(t) - \int \frac{tdt}{1+t^2} + \int \frac{tdt}{1+t^2}\right]$$

$$= 2t\, tan^{-1}(t)$$

$$= 2\ tan\, {}^{x}/_{2}\ tan^{-1}(\ tan\, {}^{x}/_{2})$$

$$= 2\, tan\, {}^{x}/_{2} \cdot {}^{x}/_{2}$$

$$\boldsymbol{\therefore \int \frac{x + \sin x}{1 + \cos x} dx = x\ tan\, {}^{x}/_{2}}$$

11.Evaluate:

$$\boldsymbol{\int \frac{x\, tan^{-1} x}{(1+x^2)^{3/2}} dx}$$

Solution:

$$Put\ x = \tan\theta$$

$$dx = sec^2\theta\ d\theta$$

$$\int \frac{x\, tan^{-1}x}{(1+x^2)^{3/2}} dx = \int \frac{tan\theta\ tan^{-1}(\tan\theta)}{(1+tan^2\theta)^{3/2}}\ sec^2\theta\ d\theta$$

$$= \int \frac{tan\,\theta \cdot \theta}{(sec^2\theta)^{3/2}}\ sec^2\theta\ d\theta \ \ = \int \frac{\theta\ tan\,\theta}{\sec\theta}\ d\theta$$

$$= \int \theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{1}\, d\theta$$

$$= \int \theta \sin\theta \, d\theta$$

$u = \theta => du = d\theta;\ \ dv = \sin\theta\, d\theta => v = -\cos\theta$

$$\int \theta \sin\theta\, d\theta = \theta(-\cos\theta) + \int \cos\theta\, d\theta$$

$$= -\theta\cos\theta + \sin\theta$$

$$\therefore \int \frac{\boldsymbol{x \tan^{-1} x}}{\boldsymbol{(1+x^2)^{3/2}}}\boldsymbol{dx = \sin\theta - \theta\cos\theta}$$

12. Evaluate:

$$\int_0^{\pi} \frac{\boldsymbol{x \tan x}}{\boldsymbol{\sec x + \tan x}}\ \boldsymbol{dx}$$

Solution:

$$let\ I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$$

$put\ \ x = \pi - x$

$$= \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x - \tan x}\, dx \qquad (since\ \tan(\pi - x) = -\tan x)$$

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{-(\sec x + \tan x)}\, dx$$

Adding 1 & 2 we get

$$2I = \pi \int_0^{\pi} \frac{tan\ x}{sec\ x + tan\ x}\ dx$$

$$2I = \pi \int_0^{\pi} \frac{sin\ x/cos\ x}{1/_{cos\ x} + \frac{sin\ x}{cos\ x}}\ dx$$

$$2I = \pi \int_0^{\pi} \frac{sin\ x}{cos\ x} \times \frac{cos\ x}{1 + sin\ x}\ dx$$

$$2I = \pi \int_0^{\pi} \frac{sin\ x}{1 + sin\ x} \times \frac{1 - sin\ x}{1 - sin\ x}\ dx$$

$$2I = \pi \int_0^{\pi} \frac{sin\ x - sin^2 x}{1 - sin^2 x}\ dx$$

$$2I = \pi \int_0^{\pi} \frac{sin\ x - sin^2 x}{cos^2 x}\ dx = \pi \int_0^{\pi} \frac{sin\ x}{cos^2 x} - \frac{sin^2 x}{cos^2 x}\ dx$$

$$2I = \pi \int_0^{\pi} \frac{sin\ x}{cos\ x \cdot cos\ x} - \frac{sin^2 x}{cos^2 x}\ dx$$

$$2I = \pi \int_0^{\pi} (sec\ x\ tan\ x - tan^2 x)\ dx$$

$$2I = \pi \left[\int_0^{\pi} sec\ x\ tan\ x\ dx - \int_0^{\pi} (sec^2 x - 1)\ \ dx\right]$$

$$2I = \pi \left[\int_0^{\pi} sec\ x\ tan\ x\ dx - \int_0^{\pi} sec^2 x dx + \int_0^{\pi} dx\right]$$

$$2I = \pi \left[sec\ x\ \int_0^{\pi} - tan\ x\ \int_0^{\pi} x\right]$$

$$2I = \pi[(sec\ \pi - seco) - (tan\ \pi - \tan o) + \pi - o]$$

$$2I = \pi[-1 - 1 + \pi] = \pi[\pi - 2]$$

$$2I = \pi[\pi - 2]$$

$\boldsymbol{I = \frac{\pi}{2}[\pi - 2]}$

13. If $\boldsymbol{I_n = \int_0^a x^n e^{-x} dx \ show\ that\ I_n - (n+2)I_{n-1} + a(n-1)I_{n-2} = 0}$

Solution:

$I_n = \int_0^a x^n e^{-x} dx$

$u = x^n => du = nx^{n-1}\,dx;\ \ dv = e^{-x}dx => v = -e^{-x}$

$= [-x^n e^{-x}]_0^a + \int_0^a e^{-x} n x^{n-1}\,dx$

$= -a^n e^{-a} + n\int_0^a e^{-x} x^{n-1}\ dx$

$I_n = -a^n e^{-a} + nI_{n-1}$

Sub n by n-1 in

$$I_{n-1} = -a^{n-1} \cdot e^{-a} + (n-1)I_{n-2}$$

Multiply ‘a’ on both sides

$$aI_{n-1} = a^1 - a^{n-1} \cdot e^{-a} + a(n-1)I_{n-2}$$

Subtract 2 from 1

$I_n - aI_{n-1} = -a^n e^{-a} + nI_{n-1} + a^n e^{-a} - a(n-1)I_{n-2}$

$I_n - aI_{n-1} - aI_{n-1} + a(n-1)I_{n-2} = 0$

$\boldsymbol{I_n + (n+a)I_{n-1} + a(n-1)I_{n-2} = 0}$

14. If $\boldsymbol{I_m = \int_0^\infty e^{-x} sin^m x\, dx\, , m = 2\ prove\ (1+m^2)I_m = m(m-1)I_{m-2}}$

Solution:

$u = sin^m x => du = m\, sin^{m-1} x\, cos\, x dx$

$\int dv = \int e^{-x} dx\ \ => v = \ -e^{-x}$

$$\int udv = uv - \int vdu$$

$$\int e^{-x} sin^m x\, dx = [sin^m x(-e^{-x})]_0^\infty + \int_0^\infty e^{-x} m\, sin^{m-1} x\, cos\, xdx$$

$$I_m = m \int_0^\infty e^{-x} m\, sin^{m-1} x\, cos\, xdx$$

$$u = sin^{m-1} x\, cos\, x$$

$$du = [(m-1) sin^{m-2} x\, cos\, x \cdot cos\, x - sin^{m-1} x\, sin\, x] dx$$

$$= [(m-1) sin^{m-2} x\, cos^2\, x - sin^m x] dx$$

$$= [(m-1) sin^{m-2} x\, (1 - sin^2 x) - sin^m x] dx$$

$$= [(m-1)[sin^{m-2} x - (sin^{m-2} x\, sin^2 x)] - sin^m x]\, dx$$

$$du = [(m-1) sin^{m-2} x - (m-1) sin^m x - sin^m x\,] dx$$

$$du = [(m-1) sin^{m-2} x - m\, sin^m x\,] dx$$

$$\int dv = \int e^{-x} dx \;\; => v = \; -e^{-x}$$

$$\int e^{-x} sin^m x\, dx$$

$$= m[-e^{-x}\, sin^{m-1} x\, cos\, x]_0^\infty - m \int_0^\infty (-e^{-x})\, [(m-1)\, sin^{m-2} x - m\, sin^m x\,] dx$$

$$I_m = 0 + m \int_0^\infty e^{-x} (m-1)\;\; sin^{m-2} x - m^2 \int_0^\infty e^{-x} sin^m x\, dx$$

$$I_m = m(m-1) I_{m-2} - m^2 I_m$$

$$I_m + m^2 I_m = m(m-1) I_{m-2}$$

$$\boldsymbol{I_m (1 + m^2) = m(m-1) I_{m-2}}$$

15. Obtain a reduction formula for

$$\int_0^{\frac{\pi}{2}} \boldsymbol{\theta \sin^n\theta \, d\theta \; n} > 1. \; \boldsymbol{P.T} \; \int_0^{\frac{\pi}{2}} \boldsymbol{\theta \sin^n\theta \, d\theta = \frac{1}{n^2} + \frac{n-1}{n} I_{n-2}}$$

Solution:

$$Un = \int_0^{\frac{\pi}{2}} \theta \sin^n\theta \, d\theta$$

$$Un = \int_0^{\frac{\pi}{2}} \theta \sin^{n-1}\theta \sin\theta \, d\theta$$

$$u = \theta \sin^{n-1}\theta => du = [\theta \sin^{n-2}\theta \cos\theta + \sin^{n-1}\theta] d\theta$$

$$\int dv = \int \sin\theta \, d\theta => -\cos\theta$$

$$Un = [\theta \sin^{n-1}\theta \, (\cos\theta)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (-\cos\theta) \, [\theta \, (n-1) \sin^{n-2}\theta \cos\theta + \sin^{n-1}\theta]$$

$$Un = 0 + \int_0^{\pi/2} (n-1) \, [\theta \sin^{n-2}\theta \cos^2\theta + \sin^{n-1}\theta \cos\theta] d\theta$$

$$Un = (n-1) \left[\int_0^{\pi/2} \theta \sin^{n-2}\theta \, (1 - \sin^2\theta) + \int_0^{\pi/2} \sin^{n-1}\theta \cos\theta \, d\theta \right]$$

$$Un = (n-1) \left[\int_0^{\pi/2} \theta \sin^{n-2}\theta \right] - (n-1) \int_0^{\pi/2} \theta \sin^{n-2}\theta \; \sin^2\theta \; d\theta + \left[\frac{\sin^n\theta}{n} \right]_0^{\frac{\pi}{2}}$$

$$= (n-1) U_{n-2} - (n-1) \int_0^{\pi/2} \theta \sin^n\theta \; d\theta + \left(\frac{1}{n} - 0 \right)$$

$$U_n = (n-1) U_{n-2} - (n-1) U_n + \frac{1}{n}$$

$$U_n + (n-1) U_n = (n-1) U_{n-2} + \frac{1}{n}$$

$$U_n + (1 + n - 1) = (n-1) U_{n-2} + \frac{1}{n}$$

$$nU_n = (n-1)U_{n-2} + \frac{1}{n}$$

$$\boldsymbol{U_n = \frac{n-1}{n}(n-1)U_{n-2} + \frac{1}{n^2}}$$

16. Evaluate:

$$\boldsymbol{\int x^3 \cos 2x \, dx}$$

Solution:

$$I_n = \frac{x^n sin\, mx}{m} + nx^{n-1}\frac{cos\, mx}{m^2} - \frac{n(n-1)}{m^2} I_{n-2}$$

$$I_n = \int x^3 \cos 2x \, dx$$

n=3, m=2 in

$$I_3 = \frac{x^3 sin\, 2x}{2} + \frac{3x^2 cos\, 2x}{4} - \frac{3(2)}{4} I_1$$

$$find\ I_1 \int x^1 \cos 2x \, dx = \int x \ \cos 2x \, dx$$

$$u = x => du = dx$$

$$\int dv = \int cos\ 2x\ dx => v = \frac{sin\ 2x}{2}$$

$$\int x \ \cos 2x \, dx = \frac{x\ sin\ 2x}{2} - \int \frac{\sin 2x}{2} dx$$

$$= \frac{x\ sin\ 2x}{2} - \frac{1}{2}\int sin\ 2x\ dx$$

$$= \frac{x\ sin\ 2x}{2} - \frac{1}{2}\left(\frac{-cos2x}{2}\right)$$

$$\int x \ \cos 2x \, dx = \frac{x\ sin\ 2x}{2} + \frac{cos2x}{4}$$

Sub 3 in 2 we've

$$I_3 = \frac{x^3 sin\, 2x}{2} + \frac{3x^2 cos\, 2x}{4} - \frac{6}{4}\left[\frac{x\, sin\, 2x}{2} + \frac{cos 2x}{4}\right]$$

$$\int \boldsymbol{x^3 \cos 2x\, dx} = \frac{\boldsymbol{x^3 sin\, 2x}}{\boldsymbol{2}} + \frac{\boldsymbol{3x^2 cos\, 2x}}{\boldsymbol{4}} - \frac{\boldsymbol{3}}{\boldsymbol{4}}\ \boldsymbol{x \sin 2x} - \frac{\boldsymbol{3}}{\boldsymbol{8}} \boldsymbol{cos\ 2x}.$$

17. Show that

$$\boldsymbol{\beta(m,n) = \beta(n,m) \&\ prove\ that\ \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx}$$

Solution:

$$\boldsymbol{(i)\ \ \beta(m,n) = \int_0^1 x^{m-1}\ (1-x)^{n-1}\ dx}$$

$x = 1 - y => dx = dy\ when\ x = 0, y = 1; x = 1, y = 0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}\ y^{n-1}\ (-dy)$$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}\ y^{n-1}\ \ dy$$

$$= \int_0^1 x^{m-1}\ (1-x)^{n-1}\ dx$$

$$\boldsymbol{\beta(m,n) = \beta(n,m)}$$

$$\boldsymbol{(ii)\ \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx}$$

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$$

$$put\ x = \frac{1}{1+y} => x = (1+y)^{-1}$$

$$dx = -(1+y)^{-2}dy$$

$$dx = \frac{-dy}{(1+y)^2}\ when\ x = 0, y = \infty;\ \ x = 1, y = 0$$

$$\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-dy}{(1+y)^2}\right)$$

$$= -\int_\infty^0 \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1+n-1+2}} \cdot y^{n-1}\ dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m+n}}\ dy$$

$$\boldsymbol{\beta(m,n) = \int_0^\infty \frac{1}{(1+x)^{m+n}}\ dx\ using\ the\ property\ \left[\int_0^a f(x)dx = \int_0^a f(y)dy\right]}$$

18. Show that $\boldsymbol{\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)}$

Solution:

WKT

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

Consider LHS

$$\beta(m+1,n) + \beta(m,n+1) = \frac{\Gamma m+1\ \Gamma n}{\Gamma m+1+n} + \frac{\Gamma n\ \Gamma m+1}{\Gamma n+m+1}$$

$$= \frac{\Gamma m+1\ \Gamma n + \Gamma m\ \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{m\Gamma m\Gamma n + n\Gamma m\Gamma n}{m+n\Gamma m+n}\ since\ \Gamma n+1 = n\Gamma n$$

$$= \frac{(m+n)\Gamma m\Gamma n}{(m+n)\Gamma m+n} = \frac{\Gamma m\Gamma n}{\Gamma m+n}$$

$$\boldsymbol{\beta(m+1,n) + \beta(m,n+1) = \beta(m,n)}$$

19. Prove .That $\boldsymbol{\Gamma\frac{1}{2} = \sqrt{\pi}}$

Solution:

WKT

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m + n}$$

$$put\ m = n = \frac{1}{2}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}+\frac{1}{2}}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1} \quad since\ \Gamma 1 = 1.$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \left(\Gamma\frac{1}{2}\right)^2$$

WKT

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\ cos^{2n-1}\theta\ d\theta$$

$$put\ m = n = \frac{1}{2}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} sin^{2\left({}^1/_2-1\right)}\theta\ cos^{2\left({}^1/_2-1\right)}\theta\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\ d\theta$$

$$= 2[\theta]_0^{\pi/2}$$

$$= 2\left({}^{\pi}/_2\right)$$

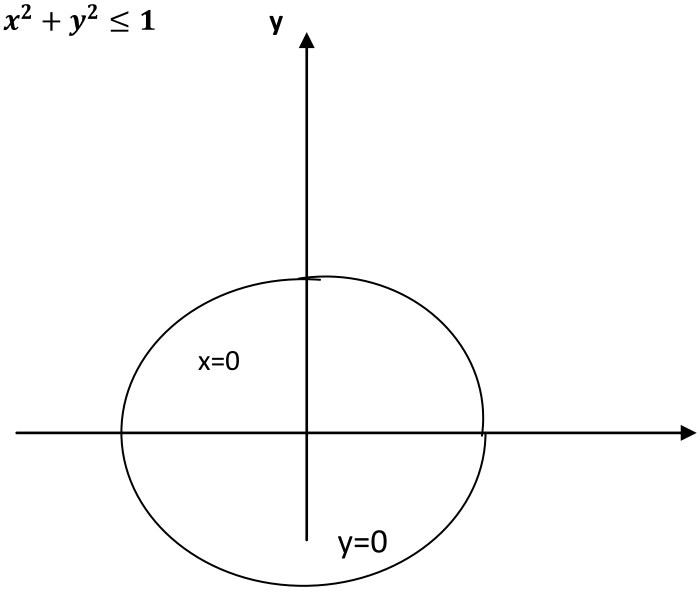
$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

From 1 & 2

$\left(\Gamma\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{2}} = \boldsymbol{\pi} => \Gamma\frac{1}{2} = \sqrt{\boldsymbol{\pi}}.$

20. Evaluate:

$\iint \boldsymbol{x^2y^2dxdy}$ ***over the circular area*** $\boldsymbol{x^2 + y^2 \leq 1}$



Solution:

$\iint x^2y^2dxdy$
x

$x \text{ varies from } 0 \text{ to } 1$

$y \text{ varies from } 0 \text{ to } \sqrt{1-x^2}$

$$\iint x^2y^2dxdy = 4\int_0^1\int_0^{\sqrt{1-x^2}} x^2y^2dxdy$$

$$= 4\int_0^1\left[\int_0^{\sqrt{1-x^2}} x^2y^2dy\right]dx$$

$$= 4\int_0^1 \left(x^2 y^3/_3\right)_0^{\sqrt{1-x^2}} dx$$

$$= 4\int_0^1 \frac{x^2\left(\sqrt{1-x^2}\right)^3}{3} dx$$

$$= \frac{4}{3}\int_0^1 x^2\,(1-x^2)^{3/_2} dx$$

Put $x = sin\theta \Rightarrow dx = cos\theta\, d\theta\ when\ x = 0, \theta = 0;\ \ x = 1, \theta = \pi/_2$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} sin^2\theta\,(1 - sin^2\theta)^{3/_2} cos\,\theta\, d\theta$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} sin^2\theta\, cos^4\,\theta\, d\theta$$

$$= \frac{4}{3}\ \frac{\Gamma\frac{2+1}{2}\Gamma\frac{4+1}{2}}{2\,\Gamma\frac{2+4+2}{2}} = \frac{\frac{4}{3}\Gamma\frac{3}{2}\Gamma\frac{5}{2}}{2\Gamma\frac{8}{2}}$$

$$= \frac{\frac{2}{3}\,1/_2\,\Gamma\frac{1}{2}\ 3/_2\,1/_2\,\Gamma\frac{1}{2}}{\Gamma 4}$$

$$= \frac{3/_2 \cdot 1/_2\sqrt{\pi}\,\sqrt{\pi}}{3\ * 3!}\ since\ \Gamma n + 1 = n! \Rightarrow\ \Gamma 3 + 1 = 3!$$

$$= \frac{{}^{3}/_{2} \cdot \pi}{3 \ * 3 * 2!} = \frac{3\pi}{72}$$

$$\iint \boldsymbol{x^2 y^2 dxdy} = \boldsymbol{\frac{3\pi}{72}}.$$

21. Evaluate $\iint xy\ dxdy$ taken over the positive of the circle

solution:

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$y = \pm\sqrt{a^2 - x^2}$$

But given positive quadrant

$\therefore y\ varies\ from\ 0\ to\ \sqrt{a^2 - x^2}$

$\therefore x\ varies\ from\ 0\ to\ a$

$$\therefore \iint xy\ dxdy = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\ dxdy$$

$$= \int_0^a x\left({}^{y^2}/_{2}\right)_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x(a^2 - x^2)\ dx$$

$$= \frac{1}{2}\int_0^a (a^2 x - x^3)\ dx = {}^{1}/_{2}\left[\frac{a^2 x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= {}^{1}/_{4}\left[a^4 - \frac{a^4}{2}\right] = {}^{1}/_{8}\,[2a^4 - a^4]$$

$$= \boldsymbol{\frac{a^4}{8}}\ .$$

PART – C

1. Show that the lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{1-z}{2}$ and 3x – 2y + z +5 = 0 and 2x + 3y + 4z – 4 = 0 lies on a plane, also. Find the point of intersection and the equation of the plane.

Solution:

To find the symmetric form of 3x – 2y + z +5 = 0 & 2x + 3y + 4z – 4=0

The equation of the plane are

$$3x - 2y + z = -5$$

$$2x + 3y + 4z = 4$$

$$put\ z = 0$$

$$3x - 2y + z = -5 \longrightarrow ①$$

$$2x + 3y + 4z = 4 \longrightarrow ②$$

Solving 1 & 2

$$1 \times 3 \;=>\; 9x - 6y = -15$$

$$2 \times 2 \;=>\; 4x + 6y = \;\; 8$$

$$x = {}^{-7}/_{13}$$

$subs \quad x = {}^{-7}/_{13}$ in ①

$3\left({}^{-7}/_{13}\right) - 2y = -5$

$\therefore y = \frac{22}{13}$

Any point on the given line is $\left({}^{-7}/_{13}, {}^{22}/_{13}, 0\right)$

Step :2

To find direction ratio's

$$3a - 2b + c = 0$$

$2a - 3b + 4c = 0$ (using the condition of perpendicular)

Solving ③ & ④

$$\frac{a}{-8-3} = \frac{b}{2-12} = \frac{c}{9+4}$$

$$\frac{a}{-11} = \frac{b}{-10} = \frac{c}{13}$$

$(a,b,c) = (-11,-10,13)$

$$\textit{The equation of the line is } \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

$$\textit{The equation of the line } 3x - 2y + z + 5 = 0 \; 2x + 3y + 4z = 4$$

$$\frac{x - {}^{7}/_{13}}{-11} = \frac{y - {}^{22}/_{13}}{-10} = \frac{z}{13}$$

Condition for co-planer

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} {}^{-7}/_{13} + 4 & {}^{22}/_{13} + 6 & 0 - 1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix} = 0$$

$$\begin{vmatrix} {}^{45}/_{13} & {}^{100}/_{13} & -1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix} = 0$$

$$\frac{45}{13}(45) \quad \frac{-100}{13}(17) \; - 1(25) = 0$$

$$\frac{2025}{13} - \frac{1700}{13} - \frac{325}{13} = 0$$

$\frac{2025}{13} - \frac{2025}{13} = 0$ => 0 = 0

The line are Co-planar

Let $\frac{x+4}{3} = \frac{y+6}{5} = \frac{1-z}{2} = r_1 \longrightarrow ⑤$

$\frac{x+{}^{7}/_{3}}{-11} = \frac{y-{}^{22}/_{13}}{-10} = \frac{z}{13} = r_2 \longrightarrow ⑥$

Any point on the line ⑤ is $(3r_1 - 4\,;\, 5r_1 - 6\,;\, 2r_1 + 1)$

Any point on the line ⑥ is $(-11r_2 - {}^{7}/_{3}\,;\, -10r_2 + {}^{22}/_{13}\,;\, 13r_2)$

Step: 3

To find the point of intersection

Comparing x terms

$3r_1 - 4 = -11r_2 - {}^{17}/_{13}$

$3r_1 + 11r_2 = -{}^{17}/_{13} + 4$

$3r_1 + 11r_2 = {}^{45}/_{13}$ ⟶ ⑦

Comparing y terms

$5r_1 - 6 = -10r_2 + {}^{22}/_{13}$

$5r_1 + 10r_2 = {}^{22}/_{13} + 6$

$5r_1 + 10r_2 = {}^{100}/_{13}$

$r_1 + 2r_2 = {}^{20}/_{13}$ ⟶ ⑧

Comparing z terms

$-2r_1 + 1 = 13r_2$

$-2r_1 - 13r_2 = -1$ ⟶ ⑨

Solving ⑦ & ⑧

⑦ $3r_1 + 11r_2 = {}^{45}/_{13}$

⑧ X 3 $3r_1 + 6r_2 = {}^{60}/_{13}$

$5r_2 \quad = {}^{-15}/_{13}$

$$\boldsymbol{r_2} = -{}^{\mathbf{3}}/_{\mathbf{13}}$$

Sub r_2 in ②

$$r_1 + 2\left({}^{-3}/_{13}\right) = \frac{20}{13}$$

$$r_1 - {}^{6}/_{13} = \frac{20}{13}$$

$$r_1 = 1$$

Sub r_1 & r_2 in ⑨

$$-2(2) - 13\left({}^{-3}/_{13}\right) = -1$$

$$-4 + 3 = -1$$

$$-1 = -1$$

The lines are intersecting

The point of intersecting is (2, 4, -3)

Equation of the plane $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

$$\begin{vmatrix} x+4 & y+6 & z-1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix} = 0$$

$$(x+4)(65-20) - (y+6)(39-22) + (z-1)(-30-55) = 0$$

$$(x+4)(45) - (y+6)(17) + (z-1)(25) = 0$$

$$\mathbf{45x - 17y + 25z + 53 = 0} \; \textit{\textbf{which is a required equation of the plane.}}$$

---

2. Find the condition that the plane lx + my + nz = 8 may touch the sphere

$$\boldsymbol{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$$

Solution:

A plane will touch a sphere if the length of from the centre of the sphere to the plane is equal to the radius of the sphere.

The centre of the given sphere is $(-u, -v, -w)$ and the radius is $\sqrt{u^2+v^2+w^2-d}$

Now the length of the perpendicular from $(-u, -v, -w)$ to

the plane $lx + my + nz + d = 0$

$$\frac{-lu - mv - nw + d}{\sqrt{l^2 + m^2 + n^2}}$$

Length of the perpendicular = radius of the sphere

$\frac{-lu-mv-nw+d}{\sqrt{l^2+m^2+n^2}}$ = $\sqrt{u^2+v^2+w^2-d}$

Squaring on both sides

$(-\boldsymbol{lu}-\boldsymbol{mv}-\boldsymbol{nw}+\boldsymbol{D})^2$= $(\boldsymbol{u}^2+\boldsymbol{v}^2+\boldsymbol{w}^2-\boldsymbol{d})(\boldsymbol{l}^2+\boldsymbol{m}^2+\boldsymbol{n}^2)$

$(\boldsymbol{lu}+\boldsymbol{mv}+\boldsymbol{nw}-\boldsymbol{D})^2$= $(\boldsymbol{l}^2+\boldsymbol{m}^2+\boldsymbol{n}^2)(\boldsymbol{u}^2+\boldsymbol{v}^2+\boldsymbol{w}^2-\boldsymbol{d})$

---

3. Find the equation of the tangent plane to the sphere $\boldsymbol{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$ at p(x, y, z) on it.

Also S.T radius to the radius through the point.

Solution:

The given sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ⟶ ①

W.K.T the line joining the centre of a sphere to any point on point it is $\perp r$ to the tangent plane at the point.

The centre of the given sphere is $c(-u,-v,-w)$ the direction ration of the line joining the point $p(x,y,z)$ *and* $c(-u,-v,-w)$ *are*

$$cp=(x_1+u, y_1+v, z_1+w)$$

The equation of the plane passing through and having cp as its normal is,

$$(x_1+u)(x-x_1)+(y_1+v)(y-y_1)+(z_1+w)(z-z_1)=0$$

$$\text{Since}\quad a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

$$xx_1+u(x-x_1)-x_1^2+yy_1+v(y-y_1)-y_1^2+zz+w(z-z_1)^2-z_1^2=0$$

$xx_1 + yy_1 + zz_1 + u(x - x_1) + v(y - y_1) + w(z - z_1) - (x_1^2 \ + y_1^2 \ + z_1^2 \ ) = \ 0 \longrightarrow$ ②

But (x_1, y_1, z_1) is a point on the sphere ① And therefore satisfies ①

(ie) $x_1^2 \ + y_1^2 \ + z_1^2 \ + \ 2ux_1 + 2vy_1 + 2wz_1 + \ d = 0$

(ie) $x_1^2 \ + y_1^2 \ + z_1^2 \ = - 2ux_1 - 2vy_1 - 2wz_1 - \ d$

Substituting the value of $x_1^2 \ + y_1^2 \ + z_1^2$ in ② we get

$$xx_1 + yy_1 + zz_1 + u(x - x_1) + v(y - y_1) + w(z - z_1) - (- 2ux_1 - 2vy_1 - 2wz_1 - \ d \ ) = 0$$

$$xx_1 + yy_1 + zz_1 + u(x - x_1) + v(y - y_1) + w(z - z_1) + (\ 2ux_1 + 2vy_1 + 2wz_1 + \ d \ ) = 0$$

$$\boldsymbol{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}$$

Which is required equations of the tangent plane.

---

4. Find theS D between z axis and straight line given by

$$\boldsymbol{ax + by + cz + d = 0 = ax + by + \ cz + d}$$

Solution :

The equation of the plane passing through the line

$ax + by + cz + d = 0 = ax + by + \ cz + d$ is

$$(ax + by + cz + d) + k(ax + by + \ cz + d) = 0$$

$(a + Ka`)x + (b + Kb`)y + (c + Kc`)z + d + Kd` = 0 \longrightarrow$ ①

The plane given by equation 1 is parallel to z axis The direction cosines of z axis (0, 0, 1)

The normal to the plane is $\perp_r$ to z axis by the condition of $\perp_r$

$$o(a + Ka`) + o(b + Kb`) + o(c + Kc`) = 0$$

$$c + Kc` = 0$$

$$Kc` = -c$$

$$K = {}^{-c}/_{c`}$$

Thus the equation of the plane is

$$\left(a - \left({}^{c}/_{c`}\right)a`\right)x + \left(b - \left({}^{c}/_{c`}\right)b`\right)y + \left(c - \left({}^{c}/_{c`}\right)c`\right)z + d - \left({}^{c}/_{c`}\right)d` = 0$$

$$(ac` - a`c)x + \left(bc` - b`c\right)y + (cc` - c`c)z + c`d - cd` = 0$$

The equation of the plane is

$$\left(ac` - a`c\right)x + \left(bc` - b`c\right)y + \left(cc` - c`c\right)z + c`d - cd` = 0 \longrightarrow ②$$

Any point on z axis is $\perp_r$ to the plane ② since (0, 0, 0) lie on z axis

The shortest distance = length of the $\perp_r$ from (0, 0, 0) to the plane ②

$$\boldsymbol{S.D = \frac{dc` - cd`}{\sqrt{(ac` - ca`)^2 + (bc` - cb`)^2}}\ Units}$$

---

5. Obtain the length the tangent from $(x_1, y_{1,} z_1)$ to the sphere

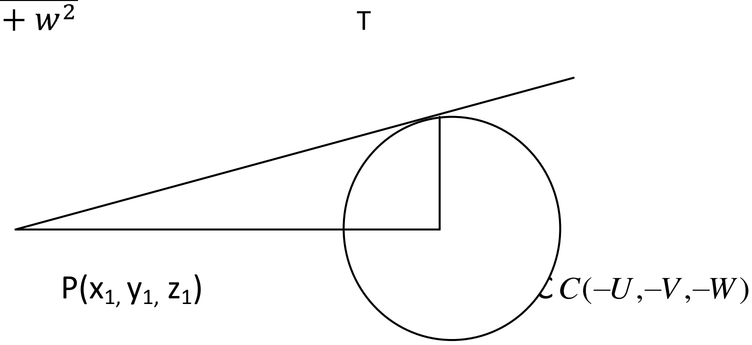
$$\boldsymbol{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0}$$

Solution: Let p be the point $(x_1, y_{1,}\ z_1)$ & C be the centre of the sphere

To prove :

A tangent from p to the sphere the co-ordinates of the sphere is equal to

$\sqrt{u^2 + v^2 + w^2}$



$$CT\ is\ \perp r\ to\ PT$$

$$PC^2 = PT^2 + CT^2$$

$$\therefore (x_1 + u)^2 + (y_1 + u)^2 + (z_1 + u)^2 = PT^2 + u^2 + v^2 + w^2 - d$$

$$\therefore \boldsymbol{PT^2 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d}$$

6. Evaluate ______________________________

$$\boldsymbol{\int_0^{\pi/4} \log(1 + tan\theta)\, d\theta}$$

Solution:

$$I = \int_0^{\pi/4} \log(1 + tan\theta) d\theta \longrightarrow ①$$

$$put\ \ \theta = {}^{\pi}/_{4} - \theta$$

$$I = \int_0^{\pi/4} \log(1 + \tan(\pi/4 - \theta)\, d\theta)$$

$$I = \int_0^{\pi/4} \log\left(1 + \frac{tan\,\pi/4 - tan\theta}{1 + tan\,\pi/4\, tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \log\left(1 + \frac{1 - tan\theta}{1 + tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \log\left(\frac{1 + tan\theta + 1 - tan\theta}{1 + tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1 + tan\theta}\right) d\theta$$

$I = \int_0^{\pi/4} [\log 2 - \log\,(1 + tan\theta)] d\theta$ ⟶ ②

From ① & ②

$2I = \int_0^{\pi/4} \log 2\, d\theta \;=>\; Log\ 2 \int_0^{\pi/4} d\theta \;=> \log 2 [\theta]_0^{\pi/4}$

$$2\,I = \log 2.\frac{\pi}{4}$$

$$I = \log 2.\frac{\pi}{8}$$

$$\therefore \int_0^{\pi/4} \log(1 + tan\theta)\, d\theta = Log\ 2.\frac{\pi}{8}$$

$$= \boldsymbol{\frac{\pi}{8} \log 2}$$

---

7. If $f(m,n) = \int_0^{\pi/2} cos^m x cos\, nx\, dx$ show that $f(m,n) = \frac{m}{m+n}\ f(m-n, n-1)$

deduce the value f(n,n)

Solution

$$f(m,n) = \int_0^{\pi/2} cos^m x cos\, nx\, dx \longrightarrow ①$$

$$u = cos^m x \qquad\qquad \int dv = \int \cos n\, x\, dx$$

$$du = m cos^{m-1} x(-\sin x) dx \qquad\qquad v = \frac{\sin nx}{n}$$

$$f(m,n) = cos^m x \frac{Sin\ nx}{n}\ \Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin nx}{n}\ m cos^{m-1} x(-\sin x) dx$$

$$= 0 + m \int_0^{\pi/2} \frac{\sin nx}{n}\ cos^{m-1} x\ \sin x\, dx$$

W.K.T

$$\cos(n-1)\, x = \cos nx\ cosx + \sin x \sin nx$$

$$\therefore \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$$

$$f(m,n) = \frac{m}{n} \int_0^{\pi/2} m cos^{m-1} x\ [\cos(n-1)\, x - \cos nx \cos x] dx$$

$$= \frac{m}{n} \int_0^{\pi/2} cos^{m-1} \cos(n-1) x\, dx - \frac{m}{n} \int_0^{\pi/2} cos^m x \cos nx\, dx$$

$f(m,n) = \frac{m}{n} f(m-1, n-1) - \frac{m}{n} f(m,n) \quad by \quad ①$

$$f(m,n) + \frac{m}{n} f(m,n) = \frac{m}{n} f(m-1, n-1)$$

$$\frac{m+n}{n} f(m,n) = \frac{m}{n} f[(m-1), (n-1)]$$

$$\boldsymbol{f(m,n) = \frac{m}{m+n} f(m-1, n-1)}$$

$$if\ m = n, then\ f(n,n) = \frac{n}{n+n}\ f(n-1, n-1)$$

$$= \frac{n}{2n}\ f(n-1, n-1) \longrightarrow ②$$

$$f(n-1, n-1) = {}^{1}/_{2}\, f(n-2, n-2) \longrightarrow ③$$

$$f(n-2, n-2) = {}^{1}/_{2}\, f(n-3, n-3) \longrightarrow ④$$

2, becomes

$$f(n,n) = {}^{1}/_{2}\, ({}^{1}/_{2})({}^{1}/_{2}) \ldots \ldots \ldots n\ factors\ f(0,0)$$

$$= \frac{1}{2^n} \int_0^{\pi/2} cos^o x \cos(ox)\, dx$$

$$= \frac{1}{2^n} \int_0^{\pi/2} dx = \frac{1}{2^n} \left[\frac{\pi}{2} - 0\right]$$

$$\boldsymbol{f(n,n) = \frac{\pi}{2^{n+1}}}$$

---

8. Evaluate

$$\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}$$

Solution:

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} \; dx \longrightarrow ①$$

$$put\ x = \left(\pi/2 - x\right)$$

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} \; dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} \; dx \longrightarrow ②$$

① + ②, $\quad 2I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

$$put\ t = \tan x/2$$

$$dt = \sec^2 x/2 \cdot 1/2\, dx \;\; => \left(1 + \tan^2 x/2\right) dx/2$$

$$dt = (1 + t^2)\, {}^{dx}\!/_{2}$$

$$dt = \frac{2dt}{1 + t^2}$$

$$when \;\; x = 0 \;\; t = 0$$

$$x = {}^{\pi}\!/_{2}, \qquad t = 1$$

$$since \;\; \sin\theta = \frac{2tan\, {}^{\theta}\!/_{2}}{1 + tan^2\, {}^{\theta}\!/_{2}}$$

$$\cos\theta = \frac{1 - tan^2\, {}^{\theta}\!/_{2}}{1 + tan^2\, {}^{\theta}\!/_{2}}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \int_0^1 \frac{2dt}{1 + t^2} \times \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$$

$$= \int_0^1 \frac{2dt}{1+t^2} \times \frac{1+t^2}{2t+1-t^2}$$

$$= 2\int_0^1 \frac{dt}{-(t^2-2t-1)}$$

$$= 2\left[\frac{1}{2\sqrt{2}}\ log\left[\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right]_0^1\right]$$

$$= \frac{2}{2\sqrt{2}}\ log\left[\frac{\sqrt{2}+(1-1)}{\sqrt{2}-(1-1)} - \frac{\sqrt{2}+(0-1)}{\sqrt{2}-(0-1)}\right]$$

$$= \frac{1}{\sqrt{2}}\ log\left[1 - \frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left(-log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \frac{1}{\sqrt{2}}\ log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\right)$$

$$= \frac{1}{\sqrt{2}}\ log\left(\frac{(\sqrt{2}-1)^2}{\left(\sqrt{2}\right)^2+1}\right) = \frac{2}{\sqrt{2}}\ log\frac{\left(\sqrt{2}+1\right)}{1}$$

$$\boldsymbol{I = \frac{1}{\sqrt{2}} log\left(\sqrt{2}+1\right) \quad since \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)}$$

9.Evaluate

$$\boldsymbol{\iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz \qquad V: x, y, z \geq 0, x+y+z \leq 1}$$

Solution

$$I = \iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz$$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\int_0^{1-x-y} z^{n-1} dz \right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\frac{Z^{n-1+1}}{n-1+1} \right]^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left(\frac{Z^n}{n} \right)^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{n} [x^{l-1} y^{m-1} (1-x-y)^n] dy\, dx$$

$$put\ y = (1-x)t\ , x - constant$$

$$dy = (1-x)dt$$

$$I = {}^1/_n \int_0^1 \int_0^{1-x} x^{l-1} [(1-x)t]^{m-1} - [1 - x(1-x)t]^n\ (1-x)dt\, dx$$

$$= {}^1/_n \int_0^1 x^{l-1} (1-x)^{(m+n+1)-1} dx \int_0^1 t^{m-1} (1-t)^{(n+1)-1} dt$$

$$= {}^1/_n\, \beta(l, m+n+1) \beta(m, n+1)$$

$$= \frac{\Gamma l\ \Gamma m + n + 1}{n \Gamma l + m + n + 1} \cdot \frac{\Gamma m \Gamma n + 1}{\Gamma m + n + 1} = \frac{\Gamma l\ \Gamma m\ \Gamma n + 1}{n \Gamma l + m + n + 1}$$

$$= \frac{\Gamma l\ \Gamma m\ \ n \Gamma n}{n \Gamma l + m + n + 1} = \frac{\Gamma m\ \Gamma l\ \Gamma n}{\Gamma l + m + n + 1}$$

Hence

$$\iiint \boldsymbol{x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz} \;=\; \frac{\boldsymbol{\Gamma\, l\, \Gamma m\;\; \Gamma n}}{\boldsymbol{\Gamma m + n + l + 1}} \;\boldsymbol{where\, Vis\, the\, region\, by}$$

$$\boldsymbol{x \geq 0, y \geq 0, z \geq 0 \; and \; x + y + z \leq 1.}$$

10. Evaluate

$$\iiint \boldsymbol{x\, y\, z\, dx\, dy\, dz \; taken\, over\, the + ve\, octant\, of\, the\, sphere\; x^2 + y^2 + z^2 = a^2}$$

Solution:

To cover the whole positive octant of the sphere $x^2 + y^2 + z^2 = a^2$

$$z \; various \; from \; 0 \; to \; \sqrt{a^2 - x^2 - y^2}$$

$$x \;\; various \; from \; 0 \; to \; \sqrt{a^2 - x^2}$$

$$y \; various \; from \; 0 \; to \;\; a$$

Hence the required integral is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \; dz\; dy\; dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\; dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{xy}{2}(a^2 - x^2 - y^2) dy\; dx$$

$$= {}^{1}\!/_{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} x(a^2y - x^2y - y^3)dy\,dx$$

$$= {}^{1}\!/_{2}\int_0^a x\left[\frac{a^2y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= {}^{1}\!/_{2}\int_0^a x\left[\frac{a^2}{2}(a^2 - x^2) - \frac{x^2}{2}(a^2 - x^2) - \frac{1}{4}(a^2 - x^2)^2\right] dx$$

$$= {}^{1}\!/_{4}\int_0^a x\left[a^4 - a^2x^2 - x^2a^2 + x^4 - {}^{1}\!/_{2}\,(a^4 + x^4 - 2a^2x^2)\right] dx$$

$$= {}^{1}\!/_{4}\int_0^a x\left[a^4 - a^2x^2 - x^2a^2 + x^4 - {}^{1}\!/_{2}\,a^4 - \frac{x^4}{2} + a^2x^2\right] dx$$

$$= {}^{1}\!/_{4}\int_0^a x\left[\frac{a^4}{2} - a^2x^2 + \frac{x^4}{2}\right] dx$$

$$= {}^{1}\!/_{8}\int_0^a [xa^4 - 2a^2x^3 + x^5]\,dx$$

$$= \frac{1}{8}\left[\frac{a^4x^2}{2} - \frac{2a^2x^4}{4} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[\frac{a^6}{2} - \frac{2a^6}{4} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\frac{3a^6 - 3a^6 + a^6}{6}\right]$$

$$= \frac{1}{8}\left(\frac{a^6}{6}\right) = \frac{a^6}{48}$$

$$\iiint x\,y\,z\,dx\,dy\,dz = \frac{a^6}{48}$$

11. Evaluate

$$\iiint x^2yz\,dx\,dy\,dz \text{ taken over the tetrahedron bounded by the planes} x = 0, y = 0, \quad z = 0 \text{ and } \frac{x}{a} + {}^{y}/_{b} + \frac{z}{c} = 1$$

Solution :

$$\text{x} = \text{y} = \text{z} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

multiply by abc

$bcx + acy - 1abz = abc \longrightarrow ①$

$put\ z = y = 0\ in$ ① $=> \ bcx = abc$

$$x = a$$

$$put \quad z = 0\ in\ ①$$

$$bcx + acy = abc$$

$$ay = ab - bx$$

$$y = \frac{ab - bx}{a}$$

$$① => \quad abz = abc - \mathrm{acy} - \mathrm{bcx}$$

$$\mathrm{z} = \frac{\mathrm{abc} - \mathrm{acy} - \mathrm{bcx}}{\mathrm{ab}}$$

$$\therefore the\ limit\ of\ x\ is\ from\ 0\ to\ a$$

$$\therefore the\ limit\ of\ y\ is\ from\ 0\ to\ \frac{ab - bx}{a}$$

$$\therefore the\ limit\ of\ z\ is\ from\ 0\ to\ \frac{abc - acy - bcx}{ab}$$

$$\iiint x^2yz\ dx\ dy\ dz = \int_0^a \int_0^{\frac{ab-bx}{a}} \int_0^{\frac{\mathrm{abc-acy-bcx}}{\mathrm{ab}}} x^2yz\ dz\ dy\ dx$$

$$= \int_0^a \int_0^{\frac{ab-bx}{a}} x^2y \left[\frac{z^2}{z}\right]_0^{\frac{\mathrm{abc-acy-bcx}}{\mathrm{ab}}} dy\ dx$$

$$= \frac{1}{2a^2b^2} \int_0^a \int_0^{\frac{ab-bx}{a}} x^2y\ [a^2b^2c^2 + a^2c^2y^2 + b^2c^2x^2 + 2abc^2xy - 2ab^2c^2x - 2a^2bc^2y]\ dydx$$

$$= \frac{c^2}{2a^2b^2} \int_0^a \int_0^{\frac{ab-bx}{a}} x^2y\ [(x-a)^2b^2 + a^2y^2 + 2ab(x-a)y]\ dydx$$

$$= \frac{c^2}{2a^2b^2} \int_0^a \int_0^{\frac{ab-bx}{a}} x^2\ [b^2(x-a)^2y + a^2y^3 + 2ab(x-a)y^2]\ dydx$$

$$= \frac{c^2}{2a^2b^2} \int_0^a x^2 \left[\frac{b^2(x-a)^2y^2}{2\times 6} + \frac{a^2y^4}{4\times 3} + \frac{2ab(x-a)y^3}{3\times 4}\right]_0^{\frac{ab-bx}{a}} dx$$

$$= \frac{c^2}{2\cdot 12a^2b^2} \int_0^a x^2 \left[6b^2(x-a)^2y^2 + 3a^2y^4 + 8ab(x-a)y^3\right]_0^{\frac{ab-bx}{a}} dx$$

$$= \frac{c^2}{24a^2b^2} \int_0^a x^2 \left[6b^2(x-a)^2 \frac{(ab-bx)^2}{a^2} + \frac{3a^2(ab-bx)^4}{a^4} + \frac{8ab(x-a)(ab-bx)^3}{a^3}\right] dx$$

$$= \frac{c^2}{24a^2b^2} \int_0^a x^2 \left[6b^2(x-a)^2 \cdot b^2(-(x-a))^2 + 3b^4(-(x-a))^4 + 8b(x-a)\cdot(-(x-a))^3\right] dx$$

$$= \frac{b^4c^2}{24a^4b^2} \int_0^a x^2 \left[6b^2(x-a)^4 + 3b^4(x-a)^4 + 8b^4(x-a)^4\right] dx$$

$$= \frac{b^2c^2}{24a^4} \int_0^a x^2 \, (x-a)^4 \, dx$$

$$= \frac{b^2c^2}{24a^4} \int_0^a x^2 \left((x^2+a^2-2ax)(x^2+a^2-2ax)\right) dx$$

$$= \frac{b^2c^2}{24a^4} \int_0^a (x^6 - 4ax^5 + 6a^2x^4 - 4a^3x^3 + a^4x^2) \, dx$$

$$= \frac{b^2c^2}{24a^4} \left[\frac{x^7}{7} - \frac{4ax^6}{6} + \frac{6a^2x^5}{5} - \frac{4a^3x^4}{4} + \frac{a^4x^3}{3}\right]_0^a$$

$$= \frac{b^2c^2}{24a^4}\left[\frac{a^7}{7} - \frac{2a^7}{3} + \frac{6a^7}{5} - a^7 + \frac{a^7}{3} - 0\right]$$

$$= \frac{b^2c^2}{24a^4}\left[\frac{5a^7 + 42a^7}{35} - a^7 + \frac{a^7}{3}\right]$$

$$= \frac{b^2c^2}{24a^4}\left[\frac{47a^7 + 35a^7}{35} - \frac{a^7}{3}\right]$$

$$= \frac{b^2c^2}{24a^4}\left[\frac{36a^7 - 35a^7}{35 \times 3}\right] = \frac{b^2c^2a^7}{2520a^4}$$

$$= \frac{a^3b^2c^2}{2520}$$

12. Expression

$\int_0^1 \boldsymbol{x^m(1 - x^n)^p dx\ interms\ of\ Gamma\ functions}$ & $\boldsymbol{evaluate\ the\ integral}$

$$\int_0^1 \boldsymbol{x^5(1 - x^3)^{10} dx}$$

Solution

$$\int_0^1 x^m(1 - x^n)^p dx$$

$put\ x^n = u\ \ x = u^{1/n}$

$$dx = {}^{1}/_{n}\, u^{1/n - 1} du = {}^{1}/_{n}\, u^{1-n/n} du\ \ When\ x = 0\ ;\ \ u = 0;\ \ x = 1;\ u = 1$$

$$\int_0^1 x^m(1-x^n)^p dx = \int_0^1 u^{m/n}(1-u)^p \left({}^1/_n u^{\frac{1-n}{n}} du \right)$$

$$= {}^1/_n \int_0^1 u^{m/n+\frac{1-n}{n}} \ (1-u)^p \ du$$

$$= {}^1/_n \int_0^1 u^{\frac{m+1-n}{n}} \ (1-u)^p \ du$$

$$= {}^1/_n \int_0^1 u^{\frac{m+1}{n}-1} \ (1-u)^{p+1-1} \ du$$

$$= {}^1/_n \int_0^1 u^{\frac{m+1}{n}-1} \ (1-u)^{(p+1)-1} \ du$$

$$Since \ \int_0^1 x^{m-1}(1-x)^{n-1} dx = \beta(m,n)$$

$$= {}^1/_n \beta\left(\frac{m+1}{n}, P+1\right)$$

Here $m = \frac{m+1}{n}$, $n = p+1$

$$\int_0^1 x^m(1-x^n)^p dx = {}^1/_n \frac{\Gamma \frac{m+1}{n} \ \Gamma p + 1}{\Gamma \frac{m+1}{n} + p + 1}$$

$$since \ \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m + n}$$

To find $\int_0^1 x^5(1-x^3)^{10} dx$

$$\int_0^1 x^5(1-x^3)^{10} dx => m = 5; n = 3; p = 10$$

$$\int_0^1 x^5(1-x^3)^{10}dx = {}^1\!/_3 \frac{\Gamma\frac{5+1}{3}\Gamma 10+1}{\Gamma\frac{5+1}{3}+10+1}$$

$$= \frac{\frac{1}{3}\Gamma 2\,\Gamma 13}{\Gamma\frac{5+1+33}{3}} = \frac{\frac{1}{3}\Gamma 2\,\Gamma 11}{\Gamma\frac{39}{3}} = \frac{\frac{1}{3}\;\Gamma\,2\,\Gamma 11}{\Gamma 13}$$

$$= \frac{\frac{1}{3}\Gamma 1+1\,\Gamma 10+1}{\Gamma 12+1} = \frac{1\,!\;\;10!}{3(12)!}$$

$$= \frac{10!}{3\times 12\times 11\times 10!} = \frac{1}{196}$$

$$\int_0^1 \boldsymbol{x^5(1-x^3)^{10}dx = \frac{1}{196}}$$

12. $\boldsymbol{Prove\ that\ \beta(m,n) = \frac{\Gamma m\ \Gamma n}{\Gamma m+n}}$

$(\boldsymbol{Or})$ $\boldsymbol{relation\ between\beta eta}$ & $\boldsymbol{\gamma\ Gamma\ function}$

Proof:

$$\Gamma m = \int_0^\infty e^{-x}x^{m-1}dx$$

$Put\ x = y^2 \quad dx = 2y \cdot dy$

$$when\, x = 0 => y = 0\,; x = \infty => y = \infty$$

$$\Gamma m = \int_0^\infty e^{-y^2}(y^2)^{m-1}\,2ydy$$

$$\Gamma m = 2\int_0^\infty e^{-y^2}(y)^{2(m-1)}\,ydy$$

$$\Gamma m = 2\int_0^\infty e^{-y^2} y^{2m-2+1} dy$$

$\Gamma m = 2\int_0^\infty e^{-y^2} y^{2m-1} dy$ ⟶ ①

Similarly $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$ ⟶ ②

Multiply ①& ②

$$\Gamma m * \Gamma n = 2\int_0^\infty e^{-y^2} y^{2m-1} dy \cdot 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

$= 4\int_0^\infty \int_0^\infty e^{-(y^2+x^2)}\, x^{2n-1}\, y^{2m-1} dxdy$ ⟶ ③

To evaluate the double integral in ③. we can use the polar coordinates put

$x = rcos\theta; y = rsine\theta$ *the area of the element dx. dy becomes* $rdrd\theta$ *to cover this region in polar co-ordinates we have to take* θ *from '0' to* $\frac{\pi}{2}$ $and\ r\ from\ 0\ to\ \infty$

$$x = r\ cos\theta \quad dxdy = rdrd\theta$$

$$y = r\ sine\theta \quad \theta = 0\ to\ {}^{\pi}/_{2}\ ; r = o\ to\ \infty$$

$$x^2 = r^2\ cos^2\theta$$

$y^2 = r^2\ sine^2\theta$ => $x^2 + y^2 = r^2$

$$\beta(m,n) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,(r\cos\theta)^{2n-1}\,(r\,\text{sine}\,\theta)^{2m-1}\, rdrd\theta$$

From ③

$$\Gamma m\,.\,\Gamma n = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,(r\cos\theta)^{2n-1}\,(r\,\text{sine}\,\theta)^{2m-1}\, rdrd\theta$$

$$= 2\int_0^{\pi/2} (r\cos\theta)^{2n-1}\,(r\,\text{sine}\,\theta)^{2m-1}\ d\theta\ *$$

$$= 2\int_0^{\infty} e^{-r^2} r^{2n-1+2m-1+1}\,dr$$

$$= \beta(m,n) * 2\int_0^{\infty} e^{-r^2} r^{2n+2m-1}\,dr$$

$$= \beta(m,n) * 2\int_0^{\infty} e^{-(x^2+y^2)} (x+y)^{2(n+m)-1}\,dxdy$$

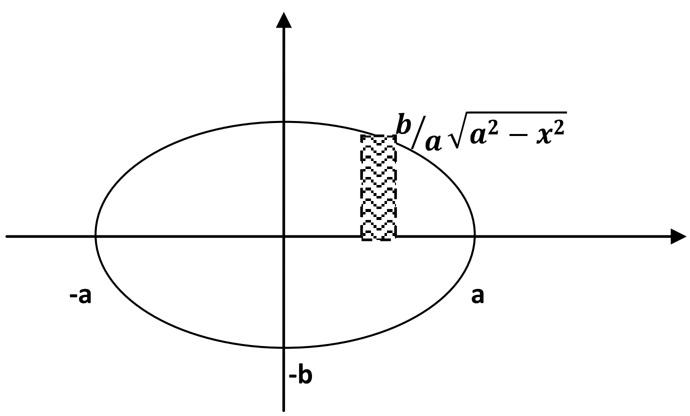
$\Gamma m\,.\,\Gamma n = \beta(m,n) * \Gamma m + n\ \ \ from$ ①&②

$$\boldsymbol{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m + n}}$$

13. Evaluate:

$\boldsymbol{\iint xy\ dxdy\ taken\ over\ positive\ quadrant\ of\ ellipse\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

Solution:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad put\ \ y = 0, \quad \frac{x^2}{a^2} = 1\,; x = \pm a$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}\ (a^2 - x^2\ )$$

$$y = \pm \frac{b}{a}\ \sqrt{a^2 - x^2}$$

But given that x,y in positive quadrant

$$\therefore x\ lies\ 0\ to\ a$$

$$\therefore y\ lies\ 0\ to\ \frac{b}{a}\ \sqrt{a^2 - x^2}$$

$$\therefore \iint xy\ dxdy = \ \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\ dxdy$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x\left(\frac{b^2}{a^2}(a^2 - x^2\ )\right) dx$$

$$= \frac{b^2}{2a^2}\int_0^a (a^2x - x^3)\ dx$$

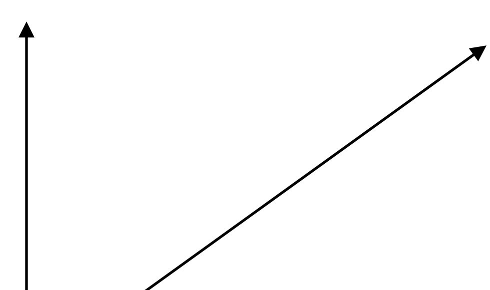
$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{b^2}{2a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

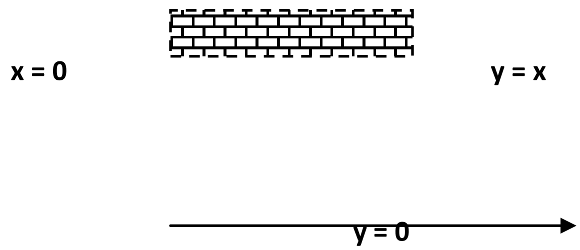
$$= \boldsymbol{\frac{a^2b^{\ 2}}{8}}$$

$$\therefore \iint \boldsymbol{xy\ dxdy\ taken\ overve\ quadrant\ of\ ellipse\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ is\ \frac{a^2b^{\ 2}}{8}}$$

14. $\boldsymbol{By\ changing\ the\ order\ of\ integration\ evaluate}$

$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\ \boldsymbol{dxdy}$





The region of integration is bounded by lines y=x, x=0 (y axis) and an infinite boundary

Take – Strips to x- axis to change the order of integration

The extremities of the strip lie on x =0 , y = x.

$\therefore$ *limits of* x *are from* 0 *to* y *and limits of* y *are from* 0 *to infinite.*

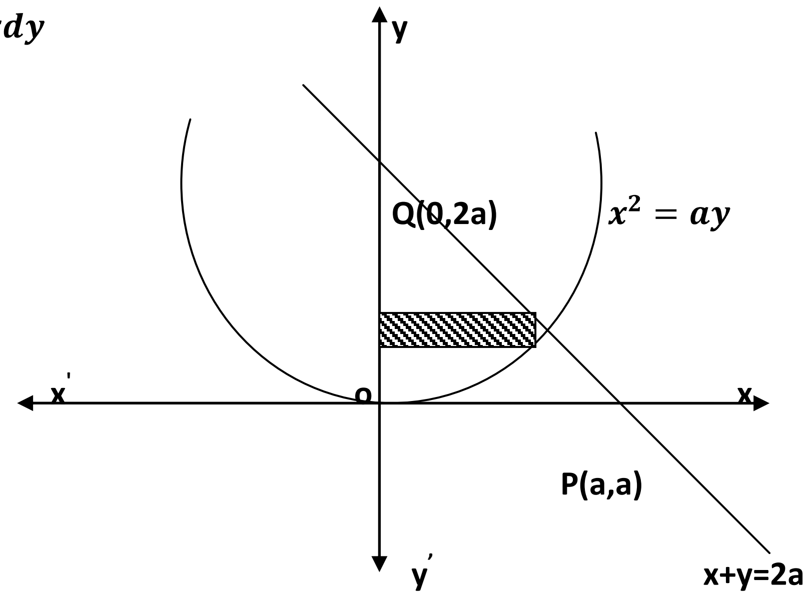
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dxdy = \int_{y=0}^\infty \int_{x=0}^y \frac{e^{-y}}{y}\, dydx$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, [x]_0^y dy = \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy$$

$$= \int_0^\infty e^{-y}\, dy = -[e^{-y}]_0^\infty = -(\mathbf{0} - \mathbf{1}) = \mathbf{1}.$$

15. $\boldsymbol{By\ changing\ the\ order\ of\ integration\ evaluate}$

$\int_0^a \int_x^{2a-x} xy \; dxdy$



Here y varies from x^2/a *to* $2a - x$ (*ie*) $y = x^2/a$ *and* $y = 2a - x$

x varies from 0 to a. hence the region of integration is OPQ .

by changing the order of integration, we first integrate x keeping y as constant. Thus the strip is parallel to x axis and x varies .

in covering the same region the end of these strips extend to the line y = 2a-x to the curve $y = x^2/a$. hence we divide the integration into two parts by the line

y=a which passes through p

hence for on region x varies from 0 *to* $\sqrt{ay}$ *and for other region* 0 *to* $(2a - y)$

in the first region y varies from 0 to a and in the next region from 0 to 2a.

$$\int_0^a \int_{x^2/a}^{2a-x} xydxdy = \int_0^a \int_0^{\sqrt{ay}} xydxdy + \int_a^{2a} \int_0^{2a-y} xydxdy$$

$x\ varies:\ \ 0\ to\ y = {a^2}/{a}\ \&\ 0\ to\ x + y = 2a => 0\ to\ x = \sqrt{ay}\ \&\ 0\ to\ x = 2a - y$

$y\ varies:\ 0\ to\ a\ \&\ x\ to\ 2a.$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_0^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{1}{2}\int_0^a y(ay)\, dy + \frac{1}{2}\int_0^{2a} y(2a-y)^2\, dy$$

$$= \frac{1}{2}\int_0^a ay^2\, dy + \frac{1}{2}\int_0^{2a} (4a^2 + y^2 - 4ay)y\ dy$$

$$= \frac{1}{2}\int_0^a ay^2\, dy + \frac{1}{2}\int_0^{2a} (4a^2y + y^3 - 4ay^2)\ dy$$

$$= \frac{1}{2}\left[\frac{ay^3}{3}\right]_0^a + \frac{1}{2}\left[\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right]_0^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[2a^2(2a)^2 + \frac{(2a)^4}{4} - \frac{4a(2a)^3}{3} - \left(2a^2 + \frac{a^4}{4} - \frac{4a^4}{3}\right)\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[10a^4 - \frac{112a^4}{3} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[\frac{120a^4 - 84a^4 - 3a^4}{12}\right]$$

$$= \mathbf{\frac{4a^4 + 5a^4}{24} = \frac{9a^4}{24} = \frac{3a^4}{8}}$$